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Phase diagram of the icosahedral and dodecahedral vector models in two dimensions

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Abstract. Fixed points of discrete vector models with icosahedral and dodecahedral symmetry are found by approximate real space renormalisation techniques. Both models have three distinct phases similar to cubic vector models. New fixed points governing the order-disorder and partial order-disorder transitions are identified. These transitions gradually become softer as the number of vector components increases.

1. Introduction

The influence of operators breaking the continuous $O(n)$ symmetry of two-dimensional spin systems to discrete subgroups has been studied intensively in the last decade. The pioneering work of José *et al* (1977) has demonstrated that breaking operators reducing the O(2) planar symmetry to $Z(p)$ are relevant for $p<5$, changing qualitatively the fixed point structure of the coupling space, and become irrelevant for $p \geqslant 5$.

Cubic anisotropies in general $O(n)$ symmetric systems were analysed by Nienhuis *et al* (1983). For $n>2$ these operators are relevant as reflected by the existence of a new cubic fixed point. As a consequence the originally trivial phase structure of the Heisenberg-type vector models is transformed into a more complicated multiphase behaviour.

In a previous publication (Margaritis *et al* 1986) we have started the study of more complicated schemes, where the O(3) symmetry is broken into the highest polyhedral groups, the icosahedron and dodecahedron. The results of numerical Hamiltonian spectrum calculations suggest the appearance of particular new fixed points characterising the order-disorder transition. The exponents of these points seem to move towards the asymptotic free low temperature behaviour of the original Heisenberg model as the rank of the residual symmetry group increases. The interest of these investigations comes from several sources. Direct Monte Carlo simulations (Berg *et al* 1984) and Monte Carlo renormalisation group studies (Hasenfratz and Margaritis 1984) provided evidence that the asymptotic free scaling laws show up already in the O(3) model for moderate coupling (temperature) $g \sim 1$, not just in the narrow neighbourhood of the critical point $g_c=0$. If the fixed points introduced by the breaking fields do not influence this region one has a chance to approximate the model by one of its polyhedral subgroups. A crossover from the 'early' asymptotic free β function (for $g\sim 1$) could occur to the final true linear behaviour close to $g \sim g_c \ll 1$. Our Hamiltonian investigation has indicated such an eventuality for the dodecahedron approximation. The advantages of the computer implementation of discrete symmetries when studying field theories on the lattice are well known (Lang *et al* 1982).

Many authors have discussed the question of universality of different lattice regularisations of the non-linear σ model (Solomon 1981, Duane and Green 1981, Sinclair 1982, Fukugita *et al* 1982). The main issue in these numerical investigations was the nature of phase transition(s) in projective realisation of the O(3) symmetry ($RP(2)$ model). It corresponds to replacing the $\boldsymbol{S}\cdot\boldsymbol{S}'$ nearest-neighbour interaction by $(\boldsymbol{S}\cdot\boldsymbol{S}')^2$ and introducing by this an extra $Z(2)$ local symmetry.

The range of conclusions is very wide. From the non-existence of any transition (Sinclair 1982) to Kosterlitz-Thouless type transitions (Solomon 1981, Fukugita *et al* 1982) one has all kinds of predictions. As was suggested by Fukugita *et al* (1982), a renormalisation group analysis (also of the polyhedral models) should settle this controversial issue. According to our analysis, in the icosahedral projective model one has a single first order transition which becomes partly second and partly first order in the case of the dodecahedron, depending on which region of the phase boundary the transition takes place in.

Certainly, we would not exclude the possibility of experimental realisations of the higher polyhedral symmetries in materials with highly degenerate ground states.

The approximate real space renormalisation group analysis to be employed in the present paper was originally proposed by Barber (1975). It is a variant in two dimensions of the Migdal-Kadanoff approximation (Migdal 1975, Kadanoff 1976). Its intuitive use by Aharony (1977) in the case of cubic vector models has led to the discovery of cubic fixed points describing a new type of phase transition in discrete spin models. The qualitative fixed point structure, but not the location of the critical points found by Aharony, proved to be correct, although the variational bond shifting method of Kadanoff (1979) as applied to the cubic vector models by Nienhuis *et al* (1983) provides a quantitatively much more reliable treatment.

The higher complexity of the icosahedral and dodecahedral models makes the implementation of the usual Migdal-Kadanoff approach somewhat more complicated. This framework allows a self-consistent treatment with only nearest-neighbour couplings, making the resulting coupling scheme and also the presentation of the renormalisation flows more transparent. Our intuitive interpretation of some quantitative fixed point characteristics follows Aharony's ideas. Also the agreement of the emerging picture with the results of finite Hamiltonian studies of the icosahedral and dodecahedral models (Margaritis *et al* 1986) strengthens our confidence in the qualitative correctness of the proposed approximation.

The organisation of the paper is as follows. In § 2 the icosahedron model is used to describe the method for exploring the fixed points of the renormalisation group equations. The actual fixed point structure and thermal indices relevant to the icosahedron model are also given in this section. For the dodecahedron model the same analysis is presented in § 3. The conclusions of these studies are summarised in § 4. In an appendix the solution of the one-dimensional transfer matrix eigenvalue problem is presented, because it is the most important input for the approximate renormalisation group equations.

2. The icosahedron model

The most general nearest-neighbour Hamiltonian with icosahedral symmetry has the form

$$-\beta H = \sum_{\langle ij \rangle} F(\boldsymbol{S}_i \cdot \boldsymbol{S}_j) \tag{2.1}$$

where $\boldsymbol{S}$ is the unit vector variable pointing to the vertices of the icosahedron and the summation refers to all nearest neighbours in the two-dimensional square lattice. The scalar product $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$ has four different values (table 1), so the Hamiltonian will be parametrised by four couplings. A convenient choice is to work with the independent Boltzmann factors $w_i = \exp(F(\boldsymbol{S}_i \cdot \boldsymbol{S}))$, where $\boldsymbol{S}_i$ ($i = 0, 1, 2, 3$) is the ith nearest direction to $\boldsymbol{S}$ in the icosahedron ($\boldsymbol{S}_0 \equiv \boldsymbol{S}$). Throughout this paper the normalisation $w_1 = \exp(F(\boldsymbol{S} \cdot \boldsymbol{S})) \equiv 1$ will be required and the actual parameter space is three dimensional:

$$-\beta H = \sum_{\langle ij \rangle} \{K[\boldsymbol{S}_i \cdot \boldsymbol{S}_j - 1] + J[(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2 - 1] + D[(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^3 - 1]\}. \tag{2.2}$$

The expression in curly brackets is one way to parametrise the most general function F up to an additive constant (which is the fourth independent parameter). This can be understood by explicitly setting up and solving the set of equations for the four independent values of F as expressed through the linear combinations of the terms in (2.2). Specifically, one finds that the relationship between the couplings K, J, D and the independent Boltzmann weights, with the help of table 1, is

$$\begin{aligned} w_2 &= \exp\left[-K\left(1-\frac{1}{\sqrt{5}}\right) - \tfrac{4}{5}J - D\left(1-\frac{1}{5\sqrt{5}}\right)\right] \\ w_3 &= \exp\left[-K\left(1+\frac{1}{\sqrt{5}}\right) - \tfrac{4}{5}J - D\left(1+\frac{1}{5\sqrt{5}}\right)\right] \\ w_4 &= \exp(-2K - 2D). \end{aligned} \tag{2.3}$$

When (2.2) is considered as a discrete approximation to the O(3)-symmetric Heisenberg model, the phase diagram in the $D \equiv 0$ plane is of particular interest because the 'linear + quadratic' Hamiltonian was studied intensively from the field theoretical point of view.

Table 1. Cosines of the angles between different directions in polyhedra.

	$\cos(\boldsymbol{n}, \boldsymbol{n}_{\text{neighbour}})$				
Polyhedron	First	Second	Third	Fourth	Fifth
Icosahedron	$1/\sqrt{5}$	$-1/\sqrt{5}$	-1	—	—
Dodecahedron	$\sqrt{5}/3$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\sqrt{5}/3$	-1

Another interesting surface is selected through the relation $w_2 = w_3$ which reduces the model to the six-component cubic model (Kim *et al* 1975, Aharony 1977). Its phase structure is well established: four fixed points were found (a six-state Potts point when $w_2 = w_3$, $w_4 = 1$, a twelve-state Potts point $w_2 = w_3 = w_4$, an Ising fixed point $w_2 = w_3 = 0$ and the cubic fixed point). The most interesting question for our study is whether any new fixed point appears when the relationship $w_2 = w_3$ is abandoned.

The real space renormalisation group method has been applied to $d = 2$ dimensional systems in various forms (Niemeyer and van Leeuwen 1976). In particular the variational bond shifting method of Kadanoff (1979) was found to yield surprisingly accurate fixed point characteristics. Our aim is a qualitative understanding of the icosahedron system. For this purpose we use the simple dedecoration RG transformation, which is a variant of the Migdal–Kadanoff iteration (Barber 1975). It provided the qualitatively correct fixed point structure of the cubic model. It can be interpreted as a minimal real space renormalisation programme on a 2×2 finite lattice with scale factor $\sqrt{2}$. Its steps are presented graphically as

The first step is the summation over the crossed variables and the transformation is completed by squaring the Boltzmann factors arising (broken line). The summation is performed in the diagonal representation of the one-dimensional icosahedral transfer matrix. The diagonalisation procedure is presented in the appendix. Four recursions for the unnormalised Boltzmann factors are arrived at. The normalisation $w_1'\equiv 1$ is ensured by dividing w_i^{new} $(i\neq 1)$ by w_1^{new}:

$$
\begin{aligned}
w_2' &= N^{-1}[a_1+3/\sqrt{5}(a_2-a_3)-a_4]\\
w_3' &= N^{-1}[a_1-3/\sqrt{5}(a_2-a_3)-a_4]\\
w_4' &= N^{-1}[a_1-3(a_2+a_3)+5a_4]
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
N &= a_1+3(a_2+a_3)+5a_4\\
a_1 &= [1+5(w_2^2+w_3^2)+w_4^2]^2\\
a_2 &= [1+\sqrt{5}(w_2^2-w_3^2)-w_4^2]^2\\
a_3 &= [1-\sqrt{5}(w_2^2-w_3^2)-w_4^2]^2\\
a_4 &= [1-w_2^2-w_3^2+w_4^2]^2.
\end{aligned}
\tag{2.5}
$$

Inspection of equations (2.4) and (2.5) reveals the invariance of the recursion equations under the interchange $w_2\leftrightarrow w_3$. This implies that it is sufficient to find eventual new fixed points beyond those of the six-state cubic model (appearing in the symmetry plane) in one-half of the three-dimensional region restricted by the positivity of the one-dimensional transfer matrix (the so-called ferromagnetic sector). In figure 1 this half region is shown with the fixed points and the phase boundaries of the model.

In the $w_2=w_3$ plane our results are identical to Aharony (1977) for the $n=6$ case. Two (with their mirror points four) additional fixed points do appear. The coordinates of these points, the eigenvalues of the recursions (2.4) and (2.5) linearised around them and the estimated thermal exponents are listed in table 2.

The icosahedron model exhibits three phases, similar to the cubic vector models. The ferromagnetic phase occurs for small w_2, w_3 and w_4 values and the (zero temperature) sink attracting this phase is S_0 of figure 1. When w_2 and w_3 are sufficiently large the system is in its paramagnetic phase, attracted by the sink S_∞. The third phase exists in the large w_4, small w_2, w_3 region, where the system exhibits partial order. All spins point along one of the diagonals of the icosahedron (with $Z(5)$ axial symmetry), but do not favour either of the directions. This region is attracted by S_{I}.

The remaining fixed points lie in the phase boundaries. Those with a single relevant direction, orthogonal to the boundaries, govern the physics of the transitions between different phases. Three leaves of the boundary contain respectively the Ising fixed

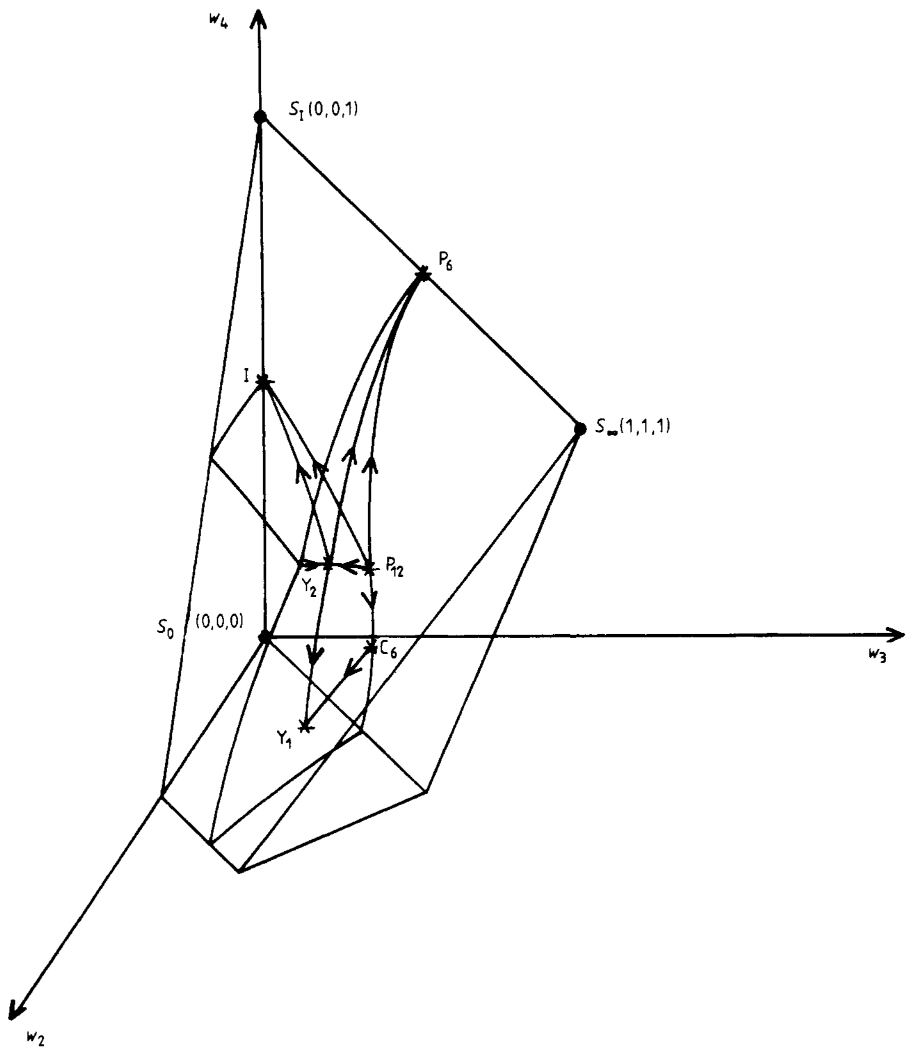

Figure 1. Schematic phase diagram of the icosahedron model. Only the $w_2 \geq w_3$ region is shown. All fixed points, except Y_1 and Y_2, are in the $w_2 = w_3$ symmetry plane.

point (I) describing the transition between partial order and order, the six-state Potts (P_6) for the transition between partial order and disorder and finally the icosahedron (Y_1) for the transition between order and disorder. Only the last one differs from the situation found in the cubic vector models.

The phase boundaries intersect in a critical line containing the unstable tricritical point Y_2 (it is attractive only along the intersection line). Two more fixed points already known from the cubic vector model are left for discussion. The cubic point C_6 describes the paramagnetic-ferromagnetic transition if the system is strictly in the $w_2 = w_3$ plane. When the iteration starts from a close-by point it approaches C_6 first and then crosses over to Y_1. P_{12} is a twelve-state Potts fixed point which is the branch point of the three phase structure in the $w_2 = w_3$ plane and naturally lies in the intersection line of the three leaves too. This point is absolutely unstable.

Table 2. Fixed points and their characteristics for the icosahedron model (asterisks denote transitions of probable first order nature).

Fixed points	Coordinates $\begin{pmatrix} w_2 \\ w_3 \\ w_4 \end{pmatrix}$	Number of relevant eigenvalues	ν
I Ising	0	1	0.669
	0		
	0.543 689		
P_6 Potts	0.427 304	1	0.514*
	0.427 304		
	1		
P_{12} Potts	0.359 529	3	0.449*
	0.359 529		
	0.359 529		
C_6 Cubic	0.363 408	1	0.450*
	0.363 408		
	0.319 122		
Y_1 Icosahedron	0.509 348	1	0.778
	0.162 785		
	0.055 979		
Y_2 Icosahedron	0.384 239	2	0.455*
	0.324 943		
	0.400 786		

Our crude RG analysis leads to distorted exponents. Following Aharony (1977) we offer an intuitive classification for the order of the transition they describe. As has been pointed out by Nienhuis and Nauenberg (1976) in the case where the largest eigenvalue of the RG transformation around a fixed point governing a transition is b^d, where b is the scale factor $(=\sqrt{2})$ and d is the dimensionality of the system, then it is a discontinuity fixed point describing a first order transition. For this reason we consider all fixed points with critical index ν close to or smaller than $\frac{1}{2}$ to belong to this class. This convention, for instance, is consistent with the exact results for the Potts model in two dimensions. By this criterion, which might be substantiated by other RG techniques, the new fixed point Y_2 corresponds to a first order transition. The Ising fixed point, which is known exactly to be second order, seems to be a convenient borderline. Then the Y_1 fixed point, whose ν exponent is larger than what is found in our approximation for the Ising point, is classified to be second order too. This conclusion confirms our previous finite-size Hamiltonian investigation of the $J=D=0$ model (Margaritis *et al* 1986) giving

$$\nu_{\text{icosa}}(\simeq\tfrac{4}{3}) > \nu_{\text{Ising}}(=1). \tag{2.6}$$

Our main result is that the first order nature of the paramagnetic–ferromagnetic transition in the cubic vector models changes into a continuous one when a larger subgroup of O(3) is chosen. The phase structure in the $D=0$ plane is displayed in figure 2. Continuous transitions are denoted by full curves, while first order ones by broken curves. Because the latent heat of the P_6 transition is known to be very small and when approaching the point B it becomes even softer, the discontinuities across

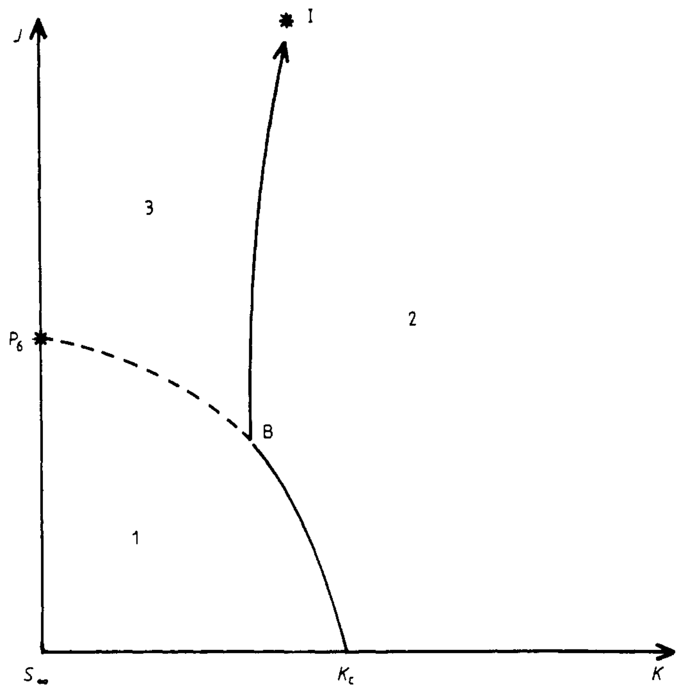

Figure 2. Schematic phase diagram for the icosahedron model on the K-J canonical surface. Continuous transitions are denoted by full curves and the first order transition by broken curves. Points on the BP_6, BI and BK_c transition lines iterate towards the I, P_6 and Y_1 fixed points, respectively. Starting from B the fixed point Y_2 is reached. The phases are: 1, disorder; 2, order; 3, partial order.

the P_6-B curve are expected to be weak. With conventional Monte Carlo simulation it might be hard to distinguish it from a continuous critical line (Hamer 1983).

Along the quadratic J axis a single first order transition is signalled. This point will be discussed further in § 4.

3. The dodecahedron model

With normalisation $w_1 \equiv 1$ the most general nearest-neighbour Hamiltonian with dodecahedron symmetry is specified by five parameters

$$-\beta H = \sum_{\langle ij \rangle} \{K[S_i \cdot S_j - 1] + J[(S_i \cdot S_j)^2 - 1] + D[(S_i \cdot S_j)^3 - 1] + E[(S_i \cdot S_j)^4 - 1] + F[(S_i \cdot S_j)^5 - 1]\}. \tag{3.1}$$

Here again the recursion equations are written in terms of the independent Boltzmann factors w_i ($i = 2, 3, 4, 5, 6$). With the help of the appendix, the recursion relations are as follows:

$$\begin{aligned}
w_2' &= N^{-1}(a_1 + \tfrac{5}{3}a_2 - \tfrac{8}{3}a_3 + \sqrt{5}a_4 - \sqrt{5}a_6) \\
w_3' &= N^{-1}(a_1 - \tfrac{5}{3}a_2 + \tfrac{2}{3}a_3 + a_4 - 2a_5 + a_6) \\
w_4' &= N^{-1}(a_1 - \tfrac{5}{3}a_2 + \tfrac{2}{3}a_3 - a_4 + 2a_5 - a_6) \\
w_5' &= N^{-1}(a_1 + \tfrac{5}{3}a_2 - \tfrac{8}{3}a_3 - \sqrt{5}a_4 + \sqrt{5}a_6)
\end{aligned} \tag{3.2}$$

$$w_6' = N^{-1}(a_1+5a_2+4a_3-3a_4-4a_5-3a_6)$$

with

$$\begin{aligned}
N &= a_1+5a_2+4a_3+3a_4+4a_5+3a_6\\
a_1 &= [1+w_6^2+3(w_2^2+w_5^2)+6(w_3^2+w_4^2)]^2\\
a_2 &= [1+w_6^2+w_2^2+w_5^2-2(w_3^2+w_4^2)]^2\\
a_3 &= [1+w_6^2-2(w_2^2+w_5^2)+w_3^2+w_4^2]^2\\
a_4 &= [1-w_6^2+\sqrt{5}(w_2^2-w_5^2)+2(w_3^2-w_4^2)]^2\\
a_5 &= [1-w_6^2-3(w_3^2-w_4^2)]^2\\
a_6 &= [1-w_6^2-\sqrt{5}(w_2^2-w_5^2)-2(w_3^2-w_4^2)]^2.
\end{aligned} \tag{3.3}$$

A summary of the fixed points found in the five-dimensional unit cube can be seen in table 3. The dodecahedron model has the same phase structure as the cubic model. The phase boundaries are four-dimensional surfaces in the five-dimensional parameter space. The coordinates of the three sinks are: S_∞ ($w_i = 1$), S_0 ($w_i = 0$) and S_I ($w_2 = \ldots = w_5 = 0$, $w_6 = 1$). The recursions (3.2) and (3.3) are symmetric under the synchronous interchanges $w_2 \leftrightarrow w_5$, $w_3 \leftrightarrow w_4$. In the invariant subspace $w_3 = w_4$, $w_2 = w_5$ one might impose more restrictive relations in order to identify special subcases.

Table 3. Fixed points and their characteristics in the dodecahedron model (asterisks denote transitions of probable first order nature).

Fixed points	Number of relevant eigenvalues	ν
I Ising	1	0.669
P_{10} Potts	2	0.464*
P_{20} Potts	1	0.412*
C_{10} Cubic	2	0.413*
J_1	1	0.651
J_2	1	0.499*
D_1	1	1.480
D_2	2	0.619
D_3	3	0.414*
D_4	2	0.413*

If $w_2 = w_3 = w_4 = w_5$ the ten-component cubic vector model arises with its familiar four fixed points (P_{10}, P_{20}, I and C_{10}). Relative to the icosahedron case only the role played by C_{10} and P_{20} (compared with C_6 and P_{12}) are interchanged. This result was already known by Aharony (1977).

In the $w_6 = 1$, $w_2 = w_5$, $w_4 = w_3$ subspace a model even under local $S \to -S$ $Z(2)$ transformations appears ($K = D = F = 0$). In the corresponding two-dimensional subspace two more fixed points were found beyond P_{10} mentioned above. J_1 is located in the $w_3 < w_2$ region, while for J_2 one has $w_2 < w_3$. Both have only one relevant direction and the thermal eigenvalues correspond to first order transitions in the region attracted by J_2 and second order ones for that attracted by J_1.

The Hamiltonian quadratic in S_i ($K = D = E = F = 0$, $J \neq 0$) is a parabolic curve in the w_2, w_3 plane ($w_3 = w_2^2$). Therefore it cannot be identified with the ten-state Potts model as it was in the case for the icosahedral symmetry. The critical point of the

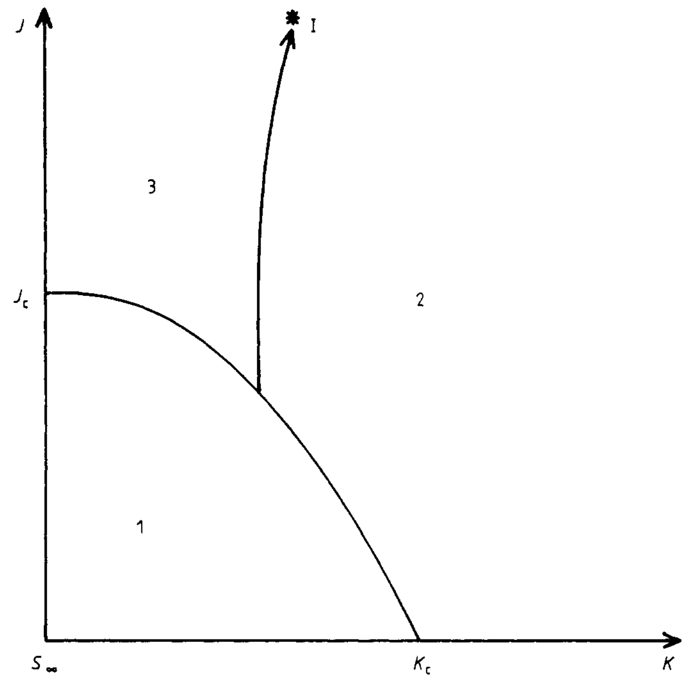

Figure 3. Schematic phase diagram for the dodecahedron model on the K–J canonical surface. All transitions are expected to be continuous. The notations of the phases are the same as in figure 2.

model is now attracted by J_1 and in consequence the disorder–partial disorder transition of the 'quadratic' model softens to a continuous one.

We have found four more fixed points if the symmetry relations $w_2 = w_5$, $w_3 = w_4$ are relaxed (D_i, $i = 1, \ldots, 4$; see table 3). Almost certainly D_i ($i \neq 1$) belong to the intersection region of the three leaves of the phase boundaries. The fixed point D_1 has a single relevant direction driving the order–disorder transition. The corresponding ν exponent is quite large, consistent with the ν hierarchy found by Margaritis *et al* (1986):

$$\nu_{\text{dodeca}} > \nu_{\text{icosa}}. \tag{3.4}$$

The projection of the phase structure into the J–K plane is shown in figure 3. All transitions are now thought to become continuous. The three branches of the critical lines are attracted by I (Ising), J_1 ('quadratic') and D_1 ('linear') fixed points, respectively. The branching point iterates towards D_2.

4. Conclusions

The main results of the present investigation are summarised below. They represent the most plausible interpretation of a numerical procedure of modest accuracy. Its application, however, to particular models (Potts or cubic vector, for instance) leads to qualitative conclusions compatible with those of more refined methods. Our confidence in the main features of the suggested phase structures are further consolidated by the agreement with our previous Hamiltonian studies realised for the specific cases $K \neq 0$, $J = D = 0$ and $K \neq 0$, $J = D = E = F = 0$, respectively.

(i) The new fixed points describing the physics of the order–disorder transition are characterised by increasing the ν exponent as the number of the allowed spin directions on the unit sphere increases.

(ii) The critical coupling K_c of the models with nearest-neighbour interactions linear in $S \cdot S'$ is shifted towards larger values, as one expects, with the approximation to O(3) becoming more faithful.

(iii) The existence of a partially ordered phase is confirmed in both models. This question was discussed for models with full O(3) symmetry by a number of authors. Solomon (1981) and Fukugita *et al* (1982) have found indications from strong coupling series and direct Monte Carlo investigations for a Kosterlitz–Thouless type transition in various models, whose interaction density was even under $S \rightarrow -S$ local transformations. Duane and Green (1981) have excluded first order transitions, while Sinclair (1982) has objected to the existence of any transition in the quadratic model.

In the icosahedron model the local $Z(2)$ invariant action is strictly quadratic in $S \cdot S'$. It is equivalent to a six-state Potts model. Our analysis has reproduced the result of Aharony and is consistent with a unique first order transition. However with conventional numerical simulation techniques on a finite lattice it might be difficult to find evidence for it because the correlation length of the six-state Potts model at the transition point is of the order of 100 lattice spacings, as one can learn from the exact Hamiltonian solution of Hamer (1983), for instance.

In the dodecahedron model the three-phase structure is still present. However the partial order–disorder transition is driven now by two new fixed points J_1, J_2. The pure quadratic model belongs to the class of second order transitions. Thus, assuming a monotonic extension of the softening tendency towards denser spin sets on the unit sphere, we also exclude the existence of a first order transition in the $RP(2)$ model.

(iv) An optimised variational bond shifting transformation could be used to find very accurate thermal exponents. As the transition we are interested in is second order, there is no need to introduce vacancies into the model (Nienhuis *et al* 1983). With a denser approximation to O(3) one might study the crossover from the predominantly quadratic ($c_2(g-g_c)^2$) regime to the linear region $(1/\nu)(g-g_c)$ of the β function and then this investigation will yield useful information for the continuum limit of the non-linear σ model.

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Appendix. Solution of one-dimensional chains with polyhedral symmetry

The Migdal–Kadanoff approximate real space renormalisation formula rests upon the exact solution of the one-dimensional spin system with the given symmetry. Moreover, the complete solution of the transfer equation

$$\sum_{S'} \exp[F(S, S')]\Psi(S') = \lambda \Psi(S) \tag{A1}$$

is an amusing problem in itself ($F(\boldsymbol{S}\cdot\boldsymbol{S}')$ is as in (2.1)). The method of solution of (A1) which will be presented in this appendix is generally applicable to any one-dimensional system with discrete symmetry.

Eigenfunctions of (A1) are fully specified by the polyhedral symmetry and do not depend on the couplings. The eigenvalues will be expressed in terms of the independent Boltzmann factors.

Consider the Hamiltonian of the icosahedron model for concreteness. The argument of Fradkin and Susskind (1978) is applicable to the limiting case $K\to\infty$, $J=D=0$: the transfer operator goes over into $1-a(K)\hat{S}$ where $a(K)$ is a unique function of K (tending to zero when $K\to\infty$) and $\hat{S}$ is the shift operator which connects a direction $\boldsymbol{n}$ in the icosahedron to its nearest neighbours.

By the above remark the eigenvectors of $\hat{S}$ coincide with those of the transfer operator for any values of the coupling. Their knowledge readily yields the eigenvalues.

The eigenvectors of $\hat{S}$ are classified with respect to their transformation properties under $Z(5)$ rotations around one of the diagonals of the icosahedron, say the '1-12' axis (figure 4). The singlet sector is spanned by applying $\hat{S}$ to appropriately chosen starting vectors

$$\Psi_0^{(\pm)}=1/\sqrt{2}(|1\rangle\pm|2\rangle) \tag{A2}$$

with definite parity under 'up-down' reflections. The Lánczos tridiagonalisation procedure (see, e.g., Wilkinson 1964) started from $\Psi_0^{(\pm)}$ closes in both channels with a 2×2 matrix:

$$P=+1:\quad \begin{pmatrix}0 & \sqrt{5}\\ \sqrt{5} & 4\end{pmatrix}\qquad P=-1:\quad \begin{pmatrix}0 & \sqrt{5}\\ \sqrt{5} & 0\end{pmatrix}. \tag{A3}$$

The corresponding eigenvectors and eigenvalues are given in table 4.

In the channels transforming non-trivially under $Z(5)$, the vectors to start the tridiagonalisation with are

$$\begin{aligned}\Psi_0^{(1)}&=\frac{1}{\sqrt{5}}(|2\rangle+Z|3\rangle+Z^2|4\rangle+Z^3|5\rangle+Z^4|6\rangle)\\ \Psi_0^{(2)}&=\frac{1}{\sqrt{5}}(|2\rangle+Z^2|3\rangle+Z^4|4\rangle+Z|5\rangle+Z^3|6\rangle)\\ Z&=\exp(\mathrm{i}2\pi/5)\end{aligned} \tag{A4}$$

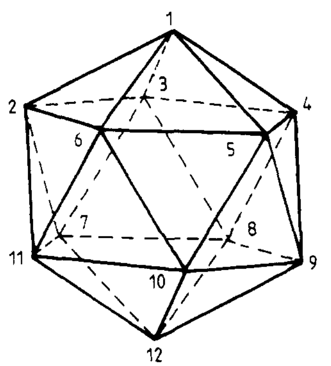

Figure 4. The canonical labelling of vertices of the icosahedron. The state when the spin points towards the ith vertex is denoted by $|i\rangle$.

Table 4. Eigenvectors and Hamiltonian eigenvalues for the icosahedron model in the $Z(5)$-invariant channel around the '1-12' diagonal (see figure 4).

α	P	Eigenvalue of $\hat{S}$	Eigenvectors
1	+	5	$1/\sqrt{12}(\|1\rangle+\ldots+\|12\rangle)$
2	−	$\sqrt{5}$	$1/2(\|1\rangle-\|12\rangle+1/\sqrt{5}(\|2\rangle+\ldots+\|11\rangle))$
3	−	$-\sqrt{5}$	$1/2(\|1\rangle-\|12\rangle-1/\sqrt{5}(\|2\rangle+\ldots+\|11\rangle))$
4	+	−1	$1/\sqrt{60}(-5(\|1\rangle+\|12\rangle)+(\|2\rangle+\ldots+\|11\rangle))$

and their complex conjugates. All four lead again to 2×2 matrices completing the specification of the eigenvectors in the full twelve-dimensional vector space. The new matrices are

$$\begin{pmatrix} Z+Z^* & 1+Z \\ 1+Z^* & Z+Z^* \end{pmatrix} \qquad \begin{pmatrix} Z^2+Z^3 & 1+Z^2 \\ 1+Z^3 & Z^2+Z^3 \end{pmatrix} \tag{A5}$$

and their respective Hermitian conjugates. The eigenvalues of the first are $\sqrt{5}$ and -1, those of the last $-\sqrt{5}$ and -1. The eigenvectors in the $Z(5)$ covariant channels are not necessary for the recursion relations explicitly. Therefore we simply conclude by stating that a singlet ($\varepsilon=5$), two three-dimensional ($\varepsilon=\pm\sqrt{5}$) and a five-dimensional representation ($\varepsilon=-1$) of the icosahedral group were found.

The Boltzmann factors can be expressed through the spectral decomposition of the transfer operator as

$$w\equiv\langle \boldsymbol{S}|\mathrm{e}^F|\boldsymbol{S}'\rangle=\sum_\alpha \lambda_\alpha\langle \boldsymbol{S}|\alpha\rangle\langle\alpha|\boldsymbol{S}'\rangle. \tag{A6}$$

For the independent quantities we have

$$w_1=\sum_\alpha \lambda_\alpha\langle 1|\alpha\rangle\langle\alpha|1\rangle \qquad w_2=\sum_\alpha \lambda_\alpha\langle 2|\alpha\rangle\langle\alpha|1\rangle$$

$$w_3=\sum_\alpha \lambda_\alpha\langle 7|\alpha\rangle\langle\alpha|1\rangle \qquad w_4=\sum_\alpha \lambda_\alpha\langle 12|\alpha\rangle\langle\alpha|1\rangle.$$

Only those $|\alpha\rangle$ contribute which have non-zero projection on $|1\rangle$. This is the reason that only the first four eigenvectors listed in table 4 do appear in (A6). The matrix relating $\boldsymbol{w}\equiv(w_1,w_2,w_3,w_4)$ and the eigenvalues of the transfer matrix $\boldsymbol{\lambda}\equiv(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is given as

$$w=\mathbf{M}\boldsymbol{\lambda} \qquad \mathbf{M}=\frac{1}{12}\begin{bmatrix} 1 & 3 & 3 & 5 \\ 1 & 3/\sqrt{5} & -3/\sqrt{5} & -1 \\ 1 & -3/\sqrt{5} & 3/\sqrt{5} & -1 \\ 1 & -3 & -3 & 5 \end{bmatrix}. \tag{A7}$$

The eigenvalues finally come by inverting (A7)

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}=\begin{pmatrix} 1 & 5 & 5 & 1 \\ 1 & \sqrt{5} & -\sqrt{5} & -1 \\ 1 & -\sqrt{5} & \sqrt{5} & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}. \tag{A8}$$

In the case of the dodecahedron one has six eigenvalues and the matrix **M** relating them to the Boltzmann factors is found in exactly the same way as for the icosahedron:

$$\mathbf{M} = \frac{1}{20}\begin{bmatrix} 1 & 5 & 4 & 3 & 4 & 3 \\ 1 & \frac{5}{3} & -\frac{8}{3} & \sqrt{5} & 0 & -\sqrt{5} \\ 1 & -\frac{5}{3} & \frac{2}{3} & 1 & -2 & 1 \\ 1 & -\frac{5}{3} & \frac{2}{3} & -1 & 2 & -1 \\ 1 & \frac{5}{3} & -\frac{8}{3} & -\sqrt{5} & 0 & \sqrt{5} \\ 1 & 5 & 4 & -3 & -4 & -3 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & 3 & 6 & 6 & 3 & 1 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & \sqrt{5} & 2 & -2 & -\sqrt{5} & -1 \\ 1 & 0 & -3 & 3 & 0 & -1 \\ 1 & \sqrt{5} & 2 & -2 & \sqrt{5} & -1 \end{bmatrix} \tag{A9}$$

In the derivation of the recursion equations (2.4) and (2.5) and (3.2) and (3.3), equations (A7)-(A9) are used, respectively.

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